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Path Model for a Level-Zero Extremal Weight Module over a Quantum Affine Algebra

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0 Introduction.

Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra over $\mathbb{Q}$ with the Cartan subalgebra $\mathfrak{h}$ and the Weyl group W. We fix an integral weight lattice $P \subset \mathfrak{h}^* := \mathrm{Hom}_{\mathbb{Q}}(\mathfrak{h}, \mathbb{Q})$ that contains all simple roots of $\mathfrak{g}$. Let $\lambda \in P$ be an integral weight. In [L1] and [L2], Littelmann introduced the notion of Lakshmibai-Seshadri paths of shape λ, which are piecewise linear, continuous maps $\pi : [0,1] \to P$ parametrized by pairs of a sequence of elements of $W\lambda$ and a sequence of rational numbers satisfying a certain condition, called the chain condition. Denote by $\mathbb{B}(\lambda)$ the set of Lakshmibai-Seshadri paths of shape λ. Littelmann proved that $\mathbb{B}(\lambda)$ has a normal crystal structure in the sense of [Kas3], and that if λ is a dominant integral weight, then the formal sum $\sum_{\pi \in \mathbb{B}(\lambda)} e(\pi(1))$ is equal to the character $\mathrm{ch}\, L(\lambda)$ of the integrable highest weight $\mathfrak{g}$-module $L(\lambda)$ of highest weight λ. Then he conjectured that $\mathbb{B}(\lambda)$ for dominant $\lambda \in P$ would be isomorphic to the crystal base of the integrable highest weight module of highest weight λ as crystals. This conjecture was affirmatively proved independently by Kashiwara [Kas4] and Joseph [J].

In [Kas2] and [Kas5], Kashiwara introduced an extremal weight module $V(\lambda)$ of extremal weight $\lambda \in P$ over the quantized universal enveloping algebra $U_q(\mathfrak{g})$ over $\mathbb{Q}(q)$, and showed that it has a crystal base $\mathcal{B}(\lambda)$. The extremal weight module is a natural generalization of an integrable highest (lowest) weight module. In fact, we know from [Kas2, §8] that if $\lambda \in P$ is dominant (resp. anti-dominant), then the extremal weight module $V(\lambda)$ is isomorphic to the integrable highest (resp. lowest) weight module of highest (resp. lowest) weight λ, and the crystal base $\mathcal{B}(\lambda)$ of $V(\lambda)$ is isomorphic to the crystal base of the integrable highest (resp. lowest) weight module as a crystal.

Now, we assume that $\mathfrak{g}$ is of affine type. Let I be the index set of the simple roots of $\mathfrak{g}$, and fix a special vertex $0 \in I$ as in [Kas5, §5.2]. In this paper, as an extension of the isomorphism theorem due to Kashiwara and Joseph, we prove that if λ is a level-zero fundamental weight $\varpi_i \in P$ for $i \in I_0 := I \setminus \{0\}$ (see [Kas5, §5.2]; note that ϖ_i is not dominant), then the connected component $\mathbb{B}_0(\varpi_i)$ of $\mathbb{B}(\varpi_i)$ containing $\pi_{\varpi_i}(t) := t\varpi_i$ is isomorphic to the crystal base $\mathcal{B}(\varpi_i)$ of the extremal weight module $V(\varpi_i)$ as crystals. Namely, we prove the following:

Theorem 1. *Assume that $\mathfrak{g}$ is of affine type. There exists a unique isomorphism $\Phi_{\varpi_i} : \mathcal{B}(\varpi_i) \xrightarrow{\sim} \mathbb{B}_0(\varpi_i)$ of crystals such that $\Phi_{\varpi_i}(u_{\varpi_i}) = \pi_{\varpi_i}$, where $u_{\varpi_i} \in \mathcal{B}(\varpi_i)$ is the unique extremal weight element of weight ϖ_i.*

Let $\mathfrak{g}_S$ be the Levi subalgebra corresponding to a proper subset S of the index set I, and let $U_q(\mathfrak{g}_S) \subset U_q(\mathfrak{g})$ be the quantized universal enveloping algebra of $\mathfrak{g}_S$. By restriction, we can regard the crystals $\mathbb{B}(\varpi_i)$ and $\mathbb{B}_0(\varpi_i)$ for $U_q(\mathfrak{g})$ as crystals for $U_q(\mathfrak{g}_S)$. We show the following branching rule for $\mathbb{B}(\varpi_i)$ and $\mathbb{B}_0(\varpi_i)$ as crystals for $U_q(\mathfrak{g}_S)$:

Theorem 2. *As crystals for $U_q(\mathfrak{g}_S)$, $\mathbb{B}(\varpi_i)$ and $\mathbb{B}_0(\varpi_i)$ decompose as follows:*

$$\mathbb{B}(\varpi_i) \cong \bigsqcup_{\substack{\pi \in \mathbb{B}(\varpi_i) \\ \pi:\ \mathfrak{g}_S\text{-dominant}}} \mathbb{B}_S(\pi(1)), \qquad \mathbb{B}_0(\varpi_i) \cong \bigsqcup_{\substack{\pi \in \mathbb{B}_0(\varpi_i) \\ \pi:\ \mathfrak{g}_S\text{-dominant}}} \mathbb{B}_S(\pi(1)).$$

where $\mathbb{B}_S(\lambda)$ is the set of Lakshmibai–Seshadri paths of shape λ for $U_q(\mathfrak{g}_S)$, and $\pi \in \mathbb{B}(\varpi_i)$ is said to be $\mathfrak{g}_S$-dominant if $(\pi(t))(\alpha_i^\vee) \geq 0$ for all $t \in [0,1]$ and $i \in S$.

We also show that the extremal weight module $V(\varpi_i)$ of extremal weight ϖ_i is completely reducible as a $U_q(\mathfrak{g}_S)$-module. Then, as an application of Theorems 1 and 2 above, we obtain the following branching rule for $V(\varpi_i)$:

Theorem 3. *The extremal weight module $V(\varpi_i)$ of extremal weight ϖ_i is completely reducible as a $U_q(\mathfrak{g}_S)$-module, and the decomposition of $V(\varpi_i)$ as a $U_q(\mathfrak{g}_S)$-module is given by:*

$$V(\varpi_i) \cong \bigoplus_{\substack{\pi \in \mathbb{B}_0(\varpi_i) \\ \pi:\ \mathfrak{g}_S\text{-dominant}}} V_S(\pi(1)),$$

where $V_S(\lambda)$ is the integrable highest weight $U_q(\mathfrak{g}_S)$-module of highest weight λ.

Assume that ϖ_i is minuscule, i.e., $\varpi_i(\alpha^\vee) \in \{\pm 1, 0\}$ for every dual real root $\alpha^\vee$ of $\mathfrak{g}$. Then we can check that $\mathbb{B}(\varpi_i)$ is connected, and hence $\mathbb{B}(\varpi_i) = \mathbb{B}_0(\varpi_i)$.

In this case, we get the following decomposition rule of Littelmann type for the concatenation $\mathbb{B}(\lambda) * \mathbb{B}(\varpi_i)$. Here we note that unlike Theorems 2 and 3, this theorem does not necessarily imply the decomposition rule for tensor products of corresponding $U_q(\mathfrak{g})$-modules.

Theorem 4. *Let λ be a dominant integral weight which is not a multiple of the null root δ of $\mathfrak{g}$, and assume that ϖ_i is minuscule. Then, the concatenation $\mathbb{B}(\lambda) * \mathbb{B}(\varpi_i)$ decomposes as follows:*

$$\mathbb{B}(\lambda) * \mathbb{B}(\varpi_i) \cong \bigsqcup_{\substack{\pi \in \mathbb{B}(\varpi_i) \\ \pi:\ \lambda\text{-dominant}}} \mathbb{B}(\lambda + \pi(1)),$$

where $\pi \in \mathbb{B}(\varpi_i)$ is said to be λ-dominant if $(\lambda + \pi(t))(\alpha_i^\vee) \geq 0$ for all $t \in [0,1]$ and $i \in I$.

Remark. The reader should compare Theorems 1 and 4 with the corresponding results [G, Theorems 1.5 and 1.6] of Greenstein for bounded modules.

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1 Preliminaries and Notation.

1.1 Quantized universal enveloping algebras. Let $A = (a_{ij})_{i,j \in I}$ be a symmetrizable generalized Cartan matrix, and $\mathfrak{g} := \mathfrak{g}(A)$ the Kac–Moody algebra over $\mathbb{Q}$ associated to the generalized Cartan matrix A. Denote by $\mathfrak{h}$ the Cartan subalgebra, by $\Pi := \{\alpha_i\}_{i \in I} \subset \mathfrak{h}^*$ and $\Pi^\vee := \{\alpha_i^\vee\}_{i \in I} \subset \mathfrak{h}$ the set of simple roots and simple coroots, and by $W = \langle r_i \mid i \in I \rangle$ the Weyl group. We take (and fix) an integral weight lattice $P \subset \mathfrak{h}^*$ such that $\alpha_i \in P$ for all $i \in I$.

Denote by $U_q(\mathfrak{g})$ the quantized universal enveloping algebra of $\mathfrak{g}$ over the field $\mathbb{Q}(q)$ of rational functions in q, and by $U_q^-(\mathfrak{g})$ (resp. $U_q^+(\mathfrak{g})$) the negative (resp. positive) part of $U_q(\mathfrak{g})$. We denote by $\widetilde{U}_q(\mathfrak{g}) = \bigoplus_{\lambda \in P} U_q(\mathfrak{g}) a_\lambda$ the modified quantized universal enveloping algebra of $\mathfrak{g}$, where a_λ is a formal element of weight λ (cf. [Kas2, §1.2]).

1.2 Affine Lie algebras. Assume that $\mathfrak{g}$ is of affine type. Let

$$\delta = \sum_{i \in I} a_i \alpha_i \in \mathfrak{h}^* \qquad \text{and} \qquad c = \sum_{i \in I} a_i^\vee \alpha_i^\vee \in \mathfrak{h} \tag{1.2.1}$$

be the null root and the canonical central element of $\mathfrak{g}$. We denote by $(\cdot\,,\,\cdot)$ the bilinear form on $\mathfrak{h}^*$, which is normalized by: $a_i^\vee = \frac{(\alpha_i,\alpha_i)}{2}a_i$ for all $i \in I$. Set $\mathfrak{h}_0^* := \bigoplus_{i\in I}\mathbb{Q}\alpha_i \subset \mathfrak{h}^*$, and let $\mathrm{cl} : \mathfrak{h}_0^* \twoheadrightarrow \mathfrak{h}_0^*/\mathbb{Q}\delta$ the canonical map from $\mathfrak{h}_0^*$ onto the quotient space $\mathfrak{h}_0^*/\mathbb{Q}\delta$. We have a bilinear form (also denoted by $(\cdot\,,\,\cdot)$) on $\mathfrak{h}_0^*/\mathbb{Q}\delta$ induced from the bilinear form $(\cdot\,,\,\cdot)$, which is positive-definite.

We take (and fix) a special vertex $0 \in I$ as in [Kas5, §5.2], and set $I_0 := I \setminus \{0\}$. For $i \in I_0$, let ϖ_i be a unique element in $\bigoplus_{i\in I_0}\mathbb{Q}\alpha_i$ such that $\varpi_i(\alpha_j^\vee) = \delta_{i,j}$ for all $j \in I_0$. Notice that $\Lambda_i := \varpi_i + a_i^\vee\Lambda_0$ is an i-th fundamental weight for $\mathfrak{g}$, where Λ_0 is a 0-th fundamental weight for $\mathfrak{g}$. So, we may assume that all the ϖ_i's are contained in the integral weight lattice P.

1.3 Crystal bases. Let $\mathcal{B}(\infty)$ be the crystal base of the negative part $U_q^-(\mathfrak{g})$ with u_∞ the highest weight element. Denote by e_i and f_i the raising and lowering Kashiwara operator on $\mathcal{B}(\infty)$, respectively, and define $\varepsilon_i : \mathcal{B}(\infty) \to \mathbb{Z}$ and $\varphi_i : \mathcal{B}(\infty) \to \mathbb{Z}$ by

$$\varepsilon_i(b) := \max\{n \geq 0 \mid e_i^n b \neq 0\}, \quad \varphi_i(b) := \varepsilon_i(b) + (\mathrm{wt}(b))(\alpha_i^\vee). \tag{1.3.1}$$

Denote by $* : \mathcal{B}(\infty) \to \mathcal{B}(\infty)$ the $*$-operation on $\mathcal{B}(\infty)$ (cf. [Kas1, Theorem 2.1.1] and [Kas3, §8.3]). We put $e_i^* := * \circ e_i \circ *$ and $f_i^* := * \circ f_i \circ *$ for each $i \in I$.

Theorem 1.3.1 (cf. [Kas1, Theorem 2.2.1]). *For each $i \in I$, there exists an embedding $\Psi_i^- : \mathcal{B}(\infty) \hookrightarrow \mathcal{B}(\infty) \otimes \mathcal{B}_i$ of crystals that maps u_∞ to $u_\infty \otimes b_i(0)$, where $\mathcal{B}_i := \{b_i(n) \mid n \in \mathbb{Z}\}$ is a crystal in* [Kas1, Example 1.2.6]. *In addition, if $b = (f_i^*)^k b_0$ for some $k \in \mathbb{Z}_{\geq 0}$ and $b_0 \in \mathcal{B}(\infty)$ such that $e_i^* b_0 = 0$, then $\Psi_i^-(b) = b_0 \otimes b_i(-k)$.*

We denote by $\mathcal{B}(-\infty)$ the crystal base of the positive part $U_q^+(\mathfrak{g})$ with $u_{-\infty}$ the lowest weight vector, and by e_i and f_i the raising and lowering Kashiwara operator on $\mathcal{B}(-\infty)$, respectively. We set

$$\varepsilon_i(b) := \varphi_i(b) - (\mathrm{wt}(b))(\alpha_i^\vee), \quad \varphi_i(b) := \max\{n \geq 0 \mid f_i^n b \neq 0\}. \tag{1.3.2}$$

We also have the $*$-operation $* : \mathcal{B}(-\infty) \to \mathcal{B}(-\infty)$ on $\mathcal{B}(-\infty)$. We can easily show that there exists an embedding $\Psi_i^+ : \mathcal{B}(-\infty) \hookrightarrow \mathcal{B}_i \otimes \mathcal{B}(-\infty)$ of crystals with properties similar to Ψ_i^- in Theorem 1.3.1.

Let $\mathcal{B}(\widetilde{U}_q(\mathfrak{g})) = \bigsqcup_{\lambda\in P}\mathcal{B}(U_q(\mathfrak{g})a_\lambda)$ be the crystal base of the modified quantized universal enveloping algebra $\widetilde{U}_q(\mathfrak{g})$ with u_λ the element of $\mathcal{B}(U_q(\mathfrak{g})a_\lambda)$ corresponding to $a_\lambda \in U_q(\mathfrak{g})a_\lambda$ (cf. [Kas2, Theorem 2.1.2]). We denote by e_i and f_i the raising and lowering Kashiwara operator on $\mathcal{B}(\widetilde{U}_q(\mathfrak{g}))$, and define $\varepsilon_i : \mathcal{B}(\widetilde{U}_q(\mathfrak{g})) \to \mathbb{Z}$ and $\varphi_i : \mathcal{B}(\widetilde{U}_q(\mathfrak{g})) \to \mathbb{Z}$ by

$$\varepsilon_i(b) := \max\{n \geq 0 \mid e_i^n b \neq 0\}, \qquad \varphi_i(b) := \max\{n \geq 0 \mid f_i^n b \neq 0\}. \tag{1.3.3}$$

We know the following theorem from [Kas2, Theorem 3.1.1].

Theorem 1.3.2. *There exists an isomorphism $\Xi_\lambda : \mathcal{B}(U_q(\mathfrak{g})a_\lambda) \xrightarrow{\sim} \mathcal{B}(\infty) \otimes \mathcal{T}_\lambda \otimes \mathcal{B}(-\infty)$ of crystals such that $\Xi_\lambda(u_\lambda) = u_\infty \otimes t_\lambda \otimes u_{-\infty}$, where $\mathcal{T}_\lambda := \{t_\lambda\}$ is a crystal consisting of a single element t_λ of weight λ* (cf. [Kas3, Example 7.3]).

We also denote by $* : \mathcal{B}(\widetilde{U}_q(\mathfrak{g})) \to \mathcal{B}(\widetilde{U}_q(\mathfrak{g}))$ the $*$-operation on $\mathcal{B}(\widetilde{U}_q(\mathfrak{g}))$ (cf. [Kas2, Theorem 4.3.2]). We know the following theorem from [Kas2, Corollary 4.3.3].

Theorem 1.3.3. *Let $b \in \mathcal{B}(U_q(\mathfrak{g})a_\lambda)$, and assume that $\Xi_\lambda(b) = b_1 \otimes t_\lambda \otimes b_2$ with $b_1 \in \mathcal{B}(\infty)$ and $b_2 \in \mathcal{B}(-\infty)$. Then, b^* is contained in $\mathcal{B}(U_q(\mathfrak{g})a_{\lambda'})$, where $\lambda' := -\lambda - \mathrm{wt}(b_1) - \mathrm{wt}(b_2)$, and $\Xi_{\lambda'}(b^*) = b_1^* \otimes t_{\lambda'} \otimes b_2^*$.*

1.4 The crystal base of an extremal weight module. Since $\mathcal{B}(\widetilde{U}_q(\mathfrak{g}))$ is a normal crystal, we can define an action of the Weyl group W on $\mathcal{B}(\widetilde{U}_q(\mathfrak{g}))$ (see [Kas2, §7.1]); for $i \in I$, we define an action of the simple reflection r_i by

$$r_i b := \begin{cases} f_i^n b & \text{if } n := (\mathrm{wt}(b))(\alpha_i^\vee) \geq 0 \\ e_i^{-n} b & \text{if } n := (\mathrm{wt}(b))(\alpha_i^\vee) \leq 0. \end{cases} \quad \text{for } b \in \mathcal{B}(\widetilde{U}_q(\mathfrak{g})). \tag{1.4.1}$$

An element $b \in \mathcal{B}(\widetilde{U}_q(\mathfrak{g}))$ is said to be extremal if the elements $\{wb\}_{w \in W} \subset \mathcal{B}(\widetilde{U}_q(\mathfrak{g}))$ satisfy the following condition for all $i \in I$:

$$\begin{aligned} &\text{if } \big(\mathrm{wt}(wb)\big)(\alpha_i^\vee) \geq 0, \text{ then } e_i(wb) = 0, \\ &\text{and if } \big(\mathrm{wt}(wb)\big)(\alpha_i^\vee) \leq 0, \text{ then } f_i(wb) = 0. \end{aligned} \tag{1.4.2}$$

For $\lambda \in P$, we define a subcrystal $\mathcal{B}(\lambda)$ of $\mathcal{B}(U_q(\mathfrak{g})a_\lambda)$ by

$$\mathcal{B}(\lambda) := \{b \in \mathcal{B}(U_q(\mathfrak{g})a_\lambda) \mid b^* \text{ is extremal}\}. \tag{1.4.3}$$

Remark that $u_\lambda \in \mathcal{B}(U_q(\mathfrak{g})a_\lambda)$ is contained in $\mathcal{B}(\lambda)$. We know from [Kas2, Proposition 8.2.2] and [Kas5, §3.1] that $\mathcal{B}(\lambda)$ is the crystal base of the extremal weight module $V(\lambda)$ of extremal weight λ over $U_q(\mathfrak{g})$.

2 Some Tools for Crystal Bases.

2.1 Multiple maps. We know the following theorem.

Theorem 2.1.1 ([Kas4, Theorem 3.2]). *Let $m \in \mathbb{Z}_{>0}$. There exists a unique injective map $S_{m,\infty} : \mathcal{B}(\infty) \hookrightarrow \mathcal{B}(\infty)$ such that for each $b \in \mathcal{B}(\infty)$ and $i \in I$, we have*

$$\mathrm{wt}(S_{m,\infty}(b)) = m\,\mathrm{wt}(b),\ \varepsilon_i(S_{m,\infty}(b)) = m\varepsilon_i(b),\ \varphi_i(S_{m,\infty}(b)) = m\varphi_i(b), \tag{2.1.1}$$

$$S_{m,\infty}(u_\infty) = u_\infty, \quad S_{m,\infty}(e_i b) = e_i^m S_{m,\infty}(b), \quad S_{m,\infty}(f_i b) = f_i^m S_{m,\infty}(b). \tag{2.1.2}$$

Proposition 2.1.2. *We set $S^*_{m,\infty} := * \circ S_{m,\infty} \circ *$. Then we have $S^*_{m,\infty} = S_{m,\infty}$ on $\mathcal{B}(\infty)$. Namely, the $*$-operation commutes with the map $S_{m,\infty} : \mathcal{B}(\infty) \hookrightarrow \mathcal{B}(\infty)$.*

The proposition above can be shown in a way similar to [NS2, Theorem 2.3.1]. Before giving a proof of the proposition, we show the following lemma.

Lemma 2.1.3. *The following diagram is commutative:*

$$\begin{array}{ccc} \mathcal{B}(\infty) & \xrightarrow{\Psi_j^-} & \mathcal{B}(\infty)\otimes\mathcal{B}_j \\ {\scriptstyle S^*_{m,\infty}}\downarrow & & \downarrow{\scriptstyle S^*_{m,\infty}\otimes S_{m,j}} \\ \mathcal{B}(\infty) & \xrightarrow{\Psi_j^-} & \mathcal{B}(\infty)\otimes\mathcal{B}_j. \end{array} \tag{2.1.3}$$

Here $S_{m,j} : \mathcal{B}_j \to \mathcal{B}_j$ is a map defined by $S_{m,j}(b_j(n)) := b_j(mn)$.

Proof. For $b \in \mathcal{B}(\infty)$, there exists $b_0 \in \mathcal{B}(\infty)$ such that $b = (f_j^*)^k b_0$ for some $k \in \mathbb{Z}_{\geq 0}$ and $e_j^* b_0 = 0$. Then, by Theorem 1.3.1, we have $\Psi_j^-(b) = b_0 \otimes b_j(-k)$, and hence

$$(S^*_\infty \otimes S_{m,j})(\Psi_j^-(b)) = S^*_\infty(b_0) \otimes b_j(-mk).$$

On the other hand, we see that $S^*_{m,\infty}(b) = (f_j^*)^{mk} S^*_{m,\infty}(b_0)$. If $e_j^* S^*_{m,\infty}(b_0) \neq 0$, then we have $\varepsilon_j(S_{m,\infty}(b_0^*)) \geq 1$. Since $\varepsilon_j(S_{m,\infty}(b)) = m\varepsilon_j(b) \in m\mathbb{Z}$ for all $b \in \mathcal{B}(\infty)$, we deduce that $\varepsilon_j(S_{m,\infty}(b_0^*)) \geq m$, and hence $(e_j^*)^m S^*_{m,\infty}(b_0) \neq 0$. However, since $e_j^* b_0 = 0$, we get $(e_j^*)^m S^*_{m,\infty}(b_0) = S^*_{m,\infty}(e_j^* b_0) = 0$, which is a contradiction. Therefore, we conclude that $e_j^* S^*_{m,\infty}(b_0) = 0$. It follows from Theorem 1.3.1 that

$$\Psi_j^-(S^*_{m,\infty}(b)) = \Psi_j^-((f_j^*)^{mk} S^*_{m,\infty}(b_0)) = S^*_{m,\infty}(b_0) \otimes b_j(-mk).$$

Hence we have $(S^*_{m,\infty} \otimes S_{m,j})(\Psi_j^-(b)) = \Psi_j^-(S^*_{m,\infty}(b))$. This completes the proof of the lemma. □

Proof of Proposition 2.1.2. We will prove that $S^*_\infty(b) = S_{m,\infty}(b)$ for $b \in \mathcal{B}(\infty)_{-\xi}$ by induction on the height $\mathrm{ht}(\xi)$ of ξ (note that $-\mathrm{wt}(b) \in \sum_{i\in I}\mathbb{Z}_{\geq 0}\alpha_i$ for all $b \in \mathcal{B}(\infty)$). If $\mathrm{ht}(\xi) = 0$, then b is the highest weight element $u_\infty \in \mathcal{B}(\infty)$, and hence the assertion is obvious.

Assume that $\mathrm{ht}(\xi) \geq 1$. Then, there exists some $i \in I$ such that $b_1 := e_i b \neq 0$. If $e_j^* b_1 = 0$ for all $j \in I$, then $b_1 = u_\infty$, and hence $b = f_i u_\infty$. Because $f_i^k u_\infty$ is a unique element of weight $-k\alpha_i$ for each $k \in \mathbb{Z}_{\geq 0}$, and $\mathrm{wt}(b^*) = \mathrm{wt}(b)$ for all $b \in \mathcal{B}(\infty)$, we deduce that $b^* = b$, and hence that

$$S^*_{m,\infty}(b) = \left(S_{m,\infty}(b^*)\right)^* = \left(S_{m,\infty}(b)\right)^* = \left(f_i^m u_\infty\right)^* = f_i^m u_\infty = S_{m,\infty}(b).$$

So, we may assume that there exists $j \in I$ such that $e_j^* b_1 \neq 0$. Let $b_2 \in \mathcal{B}(\infty)$ be such that $e_j^* b_2 = 0$ and $b_1 = (f_j^*)^k b_2$ for some $k \in \mathbb{Z}_{\geq 1}$. Namely, $b = f_i (f_j^*)^k b_2$ for some $k \geq 1$ and $b_2 \in \mathcal{B}(\infty)$ such that $e_j^* b_2 = 0$.

Case 1 : $i \neq j$. We show that $\Psi_j^-(S^*_{m,\infty}(b)) = \Psi_j^-(S_{m,\infty}(b))$ (recall that $\Psi_j^- : \mathcal{B}(\infty) \hookrightarrow \mathcal{B}(\infty) \otimes \mathcal{B}_j$ is an embedding of crystals). We have

$$\begin{aligned}\Psi_j^-(b) &= \Psi_j^-(f_i(f_j^*)^k b_2) = f_i \Psi_j^-((f_j^*)^k b_2) = f_i(b_2 \otimes b_j(-k)) \\ &= f_i b_2 \otimes b_j(-k).\end{aligned}$$

Here the last equality immediately follows from the definition of the tensor product of crystals (see, for example, [Kas3, §7.3]) and the condition that $i \neq j$. Therefore, we obtain

$$\begin{aligned}\Psi_j^-(S^*_{m,\infty}(b)) &= (S^*_{m,\infty} \otimes S_{m,j})(\Psi_j^-(b)) \quad \text{by Lemma 2.1.3} \\ &= S^*_{m,\infty}(f_i b_2) \otimes b_j(-mk) \\ &= S_{m,\infty}(f_i b_2) \otimes b_j(-mk) \quad \text{by the inductive assumption} \\ &= f_i^m S_{m,\infty}(b_2) \otimes b_j(-mk).\end{aligned}$$

On the other hand,

$$\begin{aligned}S_{m,\infty}(b) &= S_{m,\infty}(f_i(f_j^*)^k b_2) = f_i^m S_{m,\infty}((f_j^*)^k b_2) \\ &= f_i^m (f_j^*)^{mk} S_{m,\infty}(b_2) \quad \text{by the inductive assumption.}\end{aligned}$$

As in the proof of Lemma 2.1.3, we deduce that $e_j^* S^*_{m,\infty}(b_2) = 0$, and hence $e_j^* S_{m,\infty}(b_2) = e_j^* S^*_{m,\infty}(b_2) = 0$ by the inductive assumption. Therefore,

$$\begin{aligned}\Psi_j^-(S_{m,\infty}(b)) &= \Psi_j^-(f_i^m (f_j^*)^{mk} S_{m,\infty}(b_2)) = f_i^m \Psi_j^-((f_j^*)^{mk} S_{m,\infty}(b_2)) \\ &= f_i^m (S_{m,\infty}(b_2) \otimes b_j(-mk)) = (f_i^m S_{m,\infty}(b_2)) \otimes b_j(-mk).\end{aligned}$$

Here the last equality immediately follows again from the definition of the tensor product of crystals and the condition that $i \neq j$. Thus, we get that $\Psi_j^-(S_{m,\infty}^*(b)) = \Psi_j^-(S_{m,\infty}(b))$, and hence $S_{m,\infty}^*(b) = S_{m,\infty}(b)$.

Case 2 : $i = j$. As in Case 1, we have $\Psi_j^-(b) = f_i(b_2 \otimes b_i(-k))$. We deduce from the definition of the tensor product of crystals that

$$\Psi_i^-(b) = f_i(b_2 \otimes b_i(-k)) = \begin{cases} f_i b_2 \otimes b_i(-k) & \text{if } \varphi_i(b_2) > k, \\ b_2 \otimes b_i(-k-1) & \text{if } \varphi_i(b_2) \leq k. \end{cases}$$

Hence, as in Case 1, we get

$$\Psi_i^-(S_{m,\infty}^*(b)) = \begin{cases} f_i^m S_{m,\infty}(b_2) \otimes b_i(-mk) & \text{if } \varphi_i(b_2) > k, \\ S_{m,\infty}(b_2) \otimes b_i(-mk-m) & \text{if } \varphi_i(b_2) \leq k. \end{cases}$$

On the other hand, in exactly the same way as in Case 1, we can show that $\Psi_i^-(S_{m,\infty}(b)) = f_i^m(S_{m,\infty}(b_2) \otimes b_i(-mk))$. Because $\varphi_i(S_{m,\infty}(b_2)) = m\varphi_i(b_2)$ by (2.1.1), we deduce from the definition of the tensor product of crystals that

$$f_i^m(S_{m,\infty}(b_2) \otimes b_i(-mk)) = \begin{cases} f_i^m S_{m,\infty}(b_2) \otimes b_i(-mk) & \text{if } \varphi_i(b_2) > k, \\ S_{m,\infty}(b_2) \otimes b_i(-mk-m) & \text{if } \varphi_i(b_2) \leq k. \end{cases}$$

Therefore, we obtain that $\Psi_i^-(S_{m,\infty}^*(b)) = \Psi_i^-(S_{m,\infty}(b))$, and hence $S_{m,\infty}^*(b) = S_{m,\infty}(b)$. Thus, we have proved the proposition. □

Remark 2.1.4. A similar result holds for the crystal base $\mathcal{B}(-\infty)$. Namely, for each $m \in \mathbb{Z}_{>0}$, there exists a unique injective map $S_{m,-\infty} : \mathcal{B}(-\infty) \hookrightarrow \mathcal{B}(-\infty)$ with properties similar to $S_{m,\infty}$ in Theorem 2.1.1, and it commutes with the $*$-operation on $\mathcal{B}(-\infty)$.

For $m \in \mathbb{Z}_{>0}$, we define an injective map $\widetilde{S}_{m,\lambda} : \mathcal{B}(U_q(\mathfrak{g})a_\lambda) \hookrightarrow \mathcal{B}(U_q(\mathfrak{g})a_{m\lambda})$ as in the following commutative diagram (cf. Theorem 1.3.2):

$$\begin{array}{ccc} \mathcal{B}(U_q(\mathfrak{g})a_\lambda) & \xrightarrow[\sim]{\Xi_\lambda} & \mathcal{B}(\infty) \otimes \mathcal{T}_\lambda \otimes \mathcal{B}(-\infty) \\ {\scriptstyle \widetilde{S}_{m,\lambda}} \downarrow & & \downarrow {\scriptstyle S_{m,\infty} \otimes \tau_{m,\lambda} \otimes S_{m,-\infty}} \\ \mathcal{B}(U_q(\mathfrak{g})a_{m\lambda}) & \xleftarrow[\sim]{\Xi_{m\lambda}^{-1}} & \mathcal{B}(\infty) \otimes \mathcal{T}_{m\lambda} \otimes \mathcal{B}(-\infty), \end{array} \tag{2.1.4}$$

where $\tau_{m,\lambda} : \mathcal{T}_\lambda \to \mathcal{T}_{m\lambda}$ is defined by $\tau_{m,\lambda}(t_\lambda) := t_{m\lambda}$. We define $\widetilde{S}_m : \widetilde{U}_q(\mathfrak{g}) \hookrightarrow \widetilde{U}_q(\mathfrak{g})$ as the direct sum of all the $\widetilde{S}_{m,\lambda}$'s.

Proposition 2.1.5. *The maps $\widetilde{S}_{m,\lambda} : \mathcal{B}(U_q(\mathfrak{g})a_\lambda) \hookrightarrow \mathcal{B}(U_q(\mathfrak{g})a_{m\lambda})$ and $\widetilde{S}_m : \mathcal{B}(\widetilde{U}_q(\mathfrak{g})) \hookrightarrow \mathcal{B}(\widetilde{U}_q(\mathfrak{g}))$ have properties similar to $S_{m,\infty}$ in Theorem 2.1.1. In addition, the map $\widetilde{S}_m$ commutes with the $*$-operation on $\mathcal{B}(\widetilde{U}_q(\mathfrak{g}))$.*

Proof. The first assertion immediately follows from Theorem 2.1.1, Remark 2.1.4, and the definition of the tensor product of crystals (see also [Kas5, Appendix B]). Let us prove the second assertion. We set $\widetilde{S}^*_m := * \circ \widetilde{S}_m \circ *$. It suffices to show the following:

Claim. Let $\lambda \in P$, and $b \in \mathcal{B}(U_q(\mathfrak{g})a_\lambda)$. Then, we have that $\widetilde{S}^*_m(b) \in \mathcal{B}(U_q(\mathfrak{g})a_{m\lambda})$, and that $\Xi_{m\lambda}(\widetilde{S}^*_m(b)) = \Xi_{m\lambda}(\widetilde{S}_m(b))$.

Assume that $\Xi_\lambda(b) = b_1 \otimes t_\lambda \otimes b_2$ with $b_1 \in \mathcal{B}(\infty)$ and $b_2 \in \mathcal{B}(-\infty)$. Then we see by the definition of $\widetilde{S}_m$ that

$$\Xi_{m\lambda}(\widetilde{S}_m(b)) = (S_{m,\infty} \otimes \tau_{m,\lambda} \otimes S_{m,-\infty})(\Xi_\lambda(b)) = S_{m,\infty}(b_1) \otimes t_{m\lambda} \otimes S_{m,-\infty}(b_2).$$

On the other hand, we know from Theorem 1.3.3 that $b^* \in \mathcal{B}(U_q(\mathfrak{g})a_{\lambda'})$ and $\Xi_{\lambda'}(b^*) = b_1^* \otimes t_{\lambda'} \otimes b_2^*$, where $\lambda' := -\lambda - \mathrm{wt}(b_1) - \mathrm{wt}(b_2)$. Hence we have

$$\Xi_{m\lambda'}(\widetilde{S}_m(b^*)) = (S_{m,\infty} \otimes \tau_{m,\lambda} \otimes S_{m,-\infty})(\Xi_{\lambda'}(b^*)) = S_{m,\infty}(b_1^*) \otimes t_{m\lambda'} \otimes S_{m,-\infty}(b_2^*).$$

We deduce again from Theorem 1.3.3 that $\widetilde{S}^*_m(b) = \big(\widetilde{S}_m(b^*)\big)^* \in \mathcal{B}(U_q(\mathfrak{g})a_{m\lambda})$, and that

$$\begin{aligned}\Xi_{m\lambda}(\widetilde{S}^*_m(b)) &= S^*_{m,\infty}(b_1) \otimes t_{m\lambda} \otimes S^*_{m,-\infty}(b_2)\\ &= S_{m,\infty}(b_1) \otimes t_{m\lambda} \otimes S_{m,-\infty}(b_2) \quad \text{by Proposition 2.1.2 and Remark 2.1.4.}\end{aligned}$$

Thus, we obtain $\Xi_{m\lambda}(\widetilde{S}^*_m(b)) = \Xi_{m\lambda}(\widetilde{S}_m(b))$, as desired. □

Theorem 2.1.6. *Let $m \in \mathbb{Z}_{>0}$. There exists an injective map $S_{m,\lambda} : \mathcal{B}(\lambda) \hookrightarrow \mathcal{B}(m\lambda)$ such that $S_{m,\lambda}(u_\lambda) = u_{m\lambda}$ and such that for each $b \in \mathcal{B}(\infty)$ and $i \in I$, we have*

$$\mathrm{wt}(S_{m,\lambda}(b)) = m\,\mathrm{wt}(b), \quad \varepsilon_i(S_{m,\lambda}(b)) = m\varepsilon_i(b), \quad \varphi_i(S_{m,\lambda}(b)) = m\varphi_i(b), \tag{2.1.5}$$

$$S_{m,\lambda}(e_i b) = e_i^m S_{m,\lambda}(b), \quad S_{m,\lambda}(f_i b) = f_i^m S_{m,\lambda}(b). \tag{2.1.6}$$

Proof. Set $S_{m,\lambda} := \widetilde{S}_m|_{\mathcal{B}(\lambda)}$. Then, it is obvious from Proposition 2.1.5 that $S_{m,\lambda}(\mathcal{B}(\lambda)) \subset \mathcal{B}(U_q(\mathfrak{g})a_{m\lambda})$. Hence we need only show that $\big(S_{m,\lambda}(b)\big)^*$ is extremal for every $b \in \mathcal{B}(\lambda)$. We can easily check that the action of the Weyl group W commutes with $S_{m,\lambda}$. So, it follows from Proposition 2.1.5 that

$$w\big((S_{m,\lambda}(b))^*\big) = wS_{m,\lambda}(b^*) = S_{m,\lambda}(wb^*) \quad \text{for all } b \in \mathcal{B}(\lambda) \text{ and } w \in W.$$

Assume that $\mathrm{wt}(b^*) = \mu$. Then we see that $\mathrm{wt}\big((S_{m,\lambda}(b))^*\big) = m\mu$. Suppose that $(w(m\mu))(\alpha_i^\vee) \geq 0$ and $e_i\big(w((S_{m,\lambda}(b))^*)\big) \neq 0$. As in the proof of Lemma 2.1.3, we deduce that $e_i^m\big(w((S_{m,\lambda}(b))^*)\big) \neq 0$. Hence we have

$$S_{m,\lambda}\big(e_i(wb^*)\big) = e_i^m S_{m,\lambda}(wb^*) = e_i^m\big(wS_{m,\lambda}(b^*)\big) = e_i^m\big(w((S_{m,\lambda}(b))^*)\big) \neq 0.$$

However, since $(w(\mu))(\alpha_i^\vee) \geq 0$ and b^* is extremal, we have $e_i(wb^*) = 0$, and hence $S_{m,\lambda}\big(e_i(wb^*)\big) = 0$, which is a contradiction. Therefore, we obtain that $e_i\big(w((S_{m,\lambda}(b))^*)\big) = 0$. Similarly, we can prove that if $(w(m\mu))(\alpha_i^\vee) \leq 0$, then $f_i\big(w((S_{m,\lambda}(b))^*)\big) = 0$. This completes the proof of the theorem. □

2.2 Embedding into tensor products. In this subsection, we assume that $\mathfrak{g}$ is an affine Lie algebra (for the notation, see §1.2). We know the following theorem from [B, §2], [N, §3] in the symmetric case, and from [BN, §4] in the nonsymmetric case.

Theorem 2.2.1. *We have an embedding $G_{m,\varpi_i} : \mathcal{B}_0(m\varpi_i) \hookrightarrow \mathcal{B}(\varpi_i)^{\otimes m}$ of crystals that maps $u_{m\varpi_i}$ to $u_{\varpi_i}^{\otimes m}$.*

Remark 2.2.2. In [BN], they take a vertex $0 \in I$ such that $a_0 = 1$ (see [BN, §2.1]). So, in the case of $A_{2\ell}^{(2)}$, the choice of the vertex 0 is different from that in [Kas5, §5.2], and hence from ours. However, this does not cause a serious problem. For details, see the comment after [BN, Theorem 2.15].

Since $\mathcal{B}(\varpi_i)$ is connected (see [Kas5, Theorem 5.5]), we see that $S_{m,\varpi_i}(\mathcal{B}(\varpi_i)) \subset \mathcal{B}_0(m\varpi_i)$. Hence we can define $\sigma_{m,\varpi_i} : \mathcal{B}(\varpi_i) \hookrightarrow \mathcal{B}(\varpi_i)^{\otimes m}$ by $\sigma_{m,\varpi_i} := G_{m,\varpi_i} \circ S_{m,\varpi_i}$ for each $m \in \mathbb{Z}_{>0}$. Remark that σ_{m,ϖ_i} has the following properties:

$$\mathrm{wt}(\sigma_{m,\varpi_i}(b)) = m\,\mathrm{wt}(b), \quad \varepsilon_j(\sigma_{m,\varpi_i}(b)) = m\varepsilon_j(b), \quad \varphi_j(\sigma_{m,\varpi_i}(b)) = m\varphi_j(b), \tag{2.2.1}$$

$$\sigma_{m,\varpi_i}(u_{\varpi_i}) = u_{\varpi_i}^{\otimes m}, \quad \sigma_{m,\varpi_i}(e_j b) = e_j^m \sigma_{m,\varpi_i}(b), \quad \sigma_{m,\varpi_i}(f_j b) = f_j^m \sigma_{m,\varpi_i}(b). \tag{2.2.2}$$

Lemma 2.2.3. *Let m, $n \in \mathbb{Z}_{>0}$. Then we have $\sigma_{mn,\varpi_i} = \sigma_{n,\varpi_i}^{\otimes m} \circ \sigma_{m,\varpi_i}$.*

Proof. Since $\mathcal{B}(\varpi_i)$ is connected, every $b \in \mathcal{B}(\varpi_i)$ is of the form

$$b = x_{j_1} x_{j_2} \cdots x_{j_k} u_{\varpi_i}$$

for some $j_1, j_2, \ldots, j_k \in I$, where x_j is either e_j or f_j. We will show by induction on k that $\sigma_{mn,\varpi_i}(b) = \sigma_{n,\varpi_i}^{\otimes m} \circ \sigma_{m,\varpi_i}(b)$ for all $b \in \mathcal{B}(\varpi_i)$. If $k = 0$, then the assertion is obvious, since $b = u_{\varpi_i}$. Assume that $k \geq 1$. We set $b' := x_{j_2} \cdots x_{j_k} u_{\varpi_i}$, and $\sigma_{m,\varpi_i}(b') =: u_1 \otimes u_2 \otimes \cdots \otimes u_m \in \mathcal{B}(\varpi_i)^{\otimes m}$. Assume that

$$\sigma_{m,\varpi_i}(b) = x_{j_1}^m \sigma_{m,\varpi_i}(b') = x_{j_1}^{k_1} u_1 \otimes x_{j_1}^{k_2} u_2 \otimes \cdots \otimes x_{j_1}^{k_m} u_m$$

for some $k_1, k_2, \ldots, k_m \in \mathbb{Z}_{\geq 0}$. Then we have

$$\sigma_{n,\varpi_i}^{\otimes m} \circ \sigma_{m,\varpi_i}(b) = x_{j_1}^{nk_1}\sigma_{n,\varpi_i}(u_1) \otimes x_{j_1}^{nk_2}\sigma_{n,\varpi_i}(u_2) \otimes \cdots \otimes x_{j_1}^{nk_m}\sigma_{n,\varpi_i}(u_m).$$

Here we remark (cf. [Kas1, Lemma 1.3.6]) that for all $u_1 \otimes u_2 \otimes \cdots \otimes u_m \in \mathcal{B}(\varpi_i)^{\otimes m}$,

$$x_j(u_1 \otimes u_2 \otimes \cdots \otimes u_m) = u_1 \otimes u_2 \otimes \cdots \otimes x_j u_l \otimes \cdots \otimes u_m$$

if and only if

$$\begin{aligned} &x_j^n(\sigma_{n,\varpi_i}(u_1) \otimes \sigma_{n,\varpi_i}(u_2) \otimes \cdots \otimes \sigma_{n,\varpi_i}(u_m)) = \\ &\qquad \sigma_{n,\varpi_i}(u_1) \otimes \sigma_{n,\varpi_i}(u_2) \otimes \cdots \otimes x_j^n\sigma_{n,\varpi_i}(u_l) \otimes \cdots \otimes \sigma_{n,\varpi_i}(u_m). \end{aligned}$$

So we obtain

$$\begin{aligned} \sigma_{n,\varpi_i}^{\otimes m} \circ \sigma_{m,\varpi_i}(b) &= x_{j_1}^{mn}\big(\sigma_{n,\varpi_i}(u_1) \otimes \sigma_{n,\varpi_i}(u_2) \otimes \cdots \otimes \sigma_{n,\varpi_i}(u_m)\big) \\ &= x_{j_1}^{mn}\big(\sigma_{n,\varpi_i}^{\otimes m} \circ \sigma_{m,\varpi_i}(b')\big). \end{aligned}$$

We see that $\sigma_{n,\varpi_i}^{\otimes m} \circ \sigma_{m,\varpi_i}(b') = \sigma_{mn,\varpi_i}(b')$ by the inductive assumption, and that $\sigma_{mn,\varpi_i}(b) = x_{j_1}^{mn}\sigma_{mn,\varpi_i}(b')$. Therefore, we obtain $\sigma_{n,\varpi_i}^{\otimes m} \circ \sigma_{m,\varpi_i}(b) = \sigma_{mn,\varpi_i}(b)$. □

For each $w \in W$, we set $u_{w\varpi_i} := wu_{\varpi_i} \in \mathcal{B}(\varpi_i)$. By [Kas5, Proposition 5.8], we see that $u_{w\lambda}$ is well-defined. We can easily show the following lemma.

Lemma 2.2.4. *For each $m \in \mathbb{Z}_{>0}$ and $w \in W$, we have $\sigma_{m,\varpi_i}(u_{w\varpi_i}) = (u_{w\varpi_i})^{\otimes m}$.*

Proposition 2.2.5. *Let $b \in \mathcal{B}(\varpi_i)$. Assume that $b = x_{j_1}x_{j_2}\cdots x_{j_k}u_{\varpi_i}$, where x_j is either e_j or f_j, and set $b_l := x_{j_l}x_{j_{l+1}}\cdots x_{i_k}u_{\varpi_i}$ for $l = 1, 2 \ldots, k+1$ (here $b_{k+1} := u_{\varpi_i}$). Then there exists sufficiently large $m \in \mathbb{Z}$ such that for every $l = 1, 2 \ldots, k+1$,*

$$\sigma_{m,\varpi_i}(b_l) = u_{w_{l,1}\varpi_i} \otimes u_{w_{l,2}\varpi_i} \otimes \cdots \otimes u_{w_{l,m}\varpi_i} \tag{2.2.3}$$

for some $w_{l,1}, w_{l,2}, \ldots, w_{l,m} \in W$.

Proof. We show the assertion by induction on k. If $k = 0$, then the assertion is obvious. Assume that $k \geq 1$. By the inductive assumption, there exists $m \in \mathbb{Z}_{>0}$ such that $\sigma_{m,\varpi_i}(b_l)$ is of the desired form for every $l = 2, \ldots, k+1$. Assume that

$$\begin{aligned} \sigma_{m,\varpi_i}(b_1) &= \sigma_{m,\varpi_i}(x_{j_1}b_2) = x_{j_1}^m\sigma_{m,\varpi_i}(b_2) \\ &= x_{j_1}^{c_1}u_{w_{2,1}\varpi_i} \otimes x_{j_1}^{c_2}u_{w_{2,2}\varpi_i} \otimes \cdots \otimes x_{j_1}^{c_m}u_{w_{2,m}\varpi_i} \end{aligned}$$

for some $c_1, c_2, \ldots, c_m \in \mathbb{Z}_{\geq 0}$. We can easily check by Lemma 2.2.4 and [Kas1, Lemma 1.3.6] that if $n_p \in \mathbb{Z}_{>0}$ satisfies the condition that $(w_{2,p}\varpi_i)(\alpha_{j_1}^\vee) \mid n_p c_p$, then $\sigma_{n_p,\varpi_i}(x_{j_1}^{c_p} u_{w_{2,p}\varpi_i}) = u_{w_1\varpi_i} \otimes u_{w_2\varpi_i} \otimes \cdots \otimes u_{w_n\varpi_i}$ for some $w_1, w_2, \ldots, w_n \in W$. Therefore, by Lemma 2.2.4, we see that there exists $N \gg 0$ (for example, put $N = \prod_{p=1}^m n_p$) such that

$$(\sigma_{N,\varpi_i})^{\otimes m} \circ \sigma_{m,\varpi_i}(b_1) = u_{w_{1,1}\varpi_i} \otimes u_{w_{1,2}\varpi_i} \otimes \cdots \otimes u_{w_{1,Nm}\varpi_i}$$

for some $w_{1,1}, w_{1,2}, \ldots, w_{1,Nm} \in W$. Furthermore, we deduce from Lemma 2.2.4 that $(\sigma_{N,\varpi_i})^{\otimes m} \circ \sigma_{m,\varpi_i}(b_l)$ is of the desired form for every $l = 2, \ldots, k+1$. It follows from Lemma 2.2.3 that $(\sigma_{N,\varpi_i})^{\otimes m} \circ \sigma_{m,\varpi_i} = \sigma_{Nm,\varpi_i}$. Thus we have proved the proposition. □

3 Preliminary Results.

3.1 Some tools for path models. A path is, by definition, a piecewise linear, continuous map $\pi : [0,1] \to \mathbb{Q} \otimes_{\mathbb{Z}} P$ such that $\pi(0) = 0$. We regard two paths π and π' as equivalent if there exist piecewise linear, nondecreasing, surjective, continuous maps $\psi, \psi' : [0,1] \to [0,1]$ (reparametrization) such that $\pi \circ \psi = \pi' \circ \psi$. We denote by $\mathbb{P}$ the set of paths (modulo reparametrization) such that $\pi(1) \in P$, and by e_i and f_i the raising and lowering root operator (see [L2, §1]). By using root operators, we can endow $\mathbb{P}$ with a normal crystal structure (see [L2, §1 and §2]); we set $\mathrm{wt}(\pi) := \pi(1)$, and define $\varepsilon_i : \mathbb{P} \to \mathbb{Z}$ and $\varphi_i : \mathbb{P} \to \mathbb{Z}$ by

$$\varepsilon_i(\pi) := \max\{n \geq 0 \mid e_i^n \pi \neq 0\}, \qquad \varphi_i(\pi) := \max\{n \geq 0 \mid f_i^n \pi \neq 0\}. \tag{3.1.1}$$

Let $\lambda \in P$ be an (arbitrary) integral weight. We denote by $\mathbb{B}(\lambda) \subset \mathbb{P}$ the set of Lakshmibai–Seshadri paths of shape λ (see [L2, §4]), and set $\pi_\lambda(t) := t\lambda \in \mathbb{B}(\lambda)$. Denote by $\mathbb{B}_0(\lambda)$ the connected component of $\mathbb{B}(\lambda)$ containing π_λ. We obtain the following lemma by [L2, Lemma 2.4].

Lemma 3.1.1. *For $\pi \in \mathbb{P}$, we define $S_m : \mathbb{P} \hookrightarrow \mathbb{P}$ by $S_m(\pi) := m\pi$, where $(m\pi)(t) := m\pi(t)$ for $t \in [0,1]$. Then we have $S_m(\mathbb{B}_0(\lambda)) = \mathbb{B}_0(m\lambda)$. In addition, the map S_m has properties similar to $S_{m,\infty}$ in Theorem 2.1.1.*

For paths $\pi_1, \pi_2 \in \mathbb{P}$, we define a concatenation $\pi_1 * \pi_2 \in \mathbb{P}$ as in [L2, §1]. Because $\pi_\lambda * \pi_\lambda * \cdots * \pi_\lambda$ (m-times) is just $\pi_{m\lambda}$ modulo reparametrization, we obtain the following lemma.

Lemma 3.1.2. *We have a canonical embedding* $G_{m,\lambda} : \mathbb{B}_0(m\lambda) \hookrightarrow \mathbb{B}(\lambda)^{*m}$ *of crystals that maps* $\pi_{m\lambda}$ *to* π_λ^{*m}, *where* $\mathbb{B}(\lambda)^{*m} := \{\pi_1 * \pi_2 * \cdots * \pi_m \mid \pi_i \in \mathbb{B}(\lambda)\}$, *and* $\pi_\lambda^{*m} := \pi_\lambda * \pi_\lambda * \cdots * \pi_\lambda \in \mathbb{B}(\lambda)^{*m}$.

By combining Lemmas 3.1.1 and 3.1.2, we get an embedding $\sigma_{m,\lambda} : \mathbb{B}_0(\lambda) \hookrightarrow \mathbb{B}(\lambda)^{*m}$ defined by $\sigma_{m,\lambda} := G_{m,\lambda} \circ S_m$. It can easily be seen that this map has properties similar to (2.2.1) and (2.2.2).

Since $\mathbb{B}(\lambda)$ is a normal crystal, we can define an action of the Weyl group W on $\mathbb{B}(\lambda)$ (cf. (1.4.1); see also [L2, Theorem 8.1]). We set $\pi_{w\lambda} := w\pi_\lambda$ for $w \in W$. Note that $(w\pi_\lambda)(t) = t(w\lambda)$ for each $w \in W$. Using [L2, Lemma 2.7], we can prove the following proposition in a way similar to Proposition 2.2.5.

Proposition 3.1.3. *Let* $\pi \in \mathbb{B}_0(\lambda)$. *Assume that* $\pi = x_{j_1}x_{j_2}\cdots x_{j_k}\pi_\lambda$, *where* x_j *is either* e_j *or* f_j, *and set* $\pi_l := x_{j_l}x_{j_{l+1}}\cdots x_{i_k}\pi_\lambda$ *for* $l = 1, 2, \ldots, k+1$ *(here* $\pi_{k+1} := \pi_\lambda$*). Then, there exists sufficiently large* $m \in \mathbb{Z}$ *such that for every* $l = 1, 2 \ldots, k+1$,

$$\sigma_{m,\lambda}(\pi_l) = \pi_{w_{l,1}\lambda} * \pi_{w_{l,2}\lambda} * \cdots * \pi_{w_{l,m}\lambda} \tag{3.1.2}$$

for some $w_{l,1}, w_{l,2}, \ldots, w_{l,m} \in W$.

3.2 Preliminary lemmas. In this subsection, $\mathfrak{g}$ is assumed to be of affine type (for the notation, see §1.2). By using [L2, Lemma 2.1 c)], we can easily show the following lemma.

Lemma 3.2.1. *Let* $i \in I_0$. *For each* $w \in W$ *and* $j \in I$, *we have* $\mathrm{wt}(\pi_{w\varpi_i}) = \mathrm{wt}(u_{w\varpi_i})$, $\varepsilon_j(\pi_{w\varpi_i}) = \varepsilon_j(u_{w\varpi_i})$, *and* $\varphi_j(\pi_{w\varpi_i}) = \varphi_j(u_{w\varpi_i})$.

It follows from [Kas1, Lemma 1.3.6], [L2, Lemma 2.7], and Lemma 3.2.1 that

$$x_j^k(u_{w_1\varpi_i} \otimes u_{w_2\varpi_i} \otimes \cdots \otimes u_{w_m\varpi_i}) = x_j^{k_1}u_{w_1\varpi_i} \otimes x_j^{k_2}u_{w_2\varpi_i} \otimes \cdots \otimes x_j^{k_m}u_{w_m\varpi_i}$$

for some $k_1, k_2, \ldots, k_m \in \mathbb{Z}_{\geq 0}$ if and only if

$$x_j^k(\pi_{w_1\varpi_i} * \pi_{w_2\varpi_i} * \cdots * \pi_{w_m\varpi_i}) = x_j^{k_1}\pi_{w_1\varpi_i} * x_j^{k_2}\pi_{w_2\varpi_i} * \cdots * x_j^{k_m}\pi_{w_m\varpi_i}$$

for every $k \in \mathbb{Z}_{\geq 0}$, $m \in \mathbb{Z}_{>0}$ and $w_1, w_2, \ldots, w_m \in W$. So, we obtain the following lemma.

Lemma 3.2.2. (1) *Let* $b = x_{j_1}x_{j_2}\cdots x_{j_k}u_{\varpi_i} \in \mathcal{B}(\varpi_i)$. *Take* $m \in \mathbb{Z}_{>0}$ *such that the assertion of Proposition 2.2.5 holds, and assume that* $\sigma_{m,\varpi_i}(b) = u_{w_1\varpi_i} \otimes$

$u_{w_2\varpi_i} \otimes \cdots \otimes u_{w_m\varpi_i}$. *Then we have* $\pi := x_{j_1}x_{j_2}\cdots x_{j_k}\pi_{\varpi_i} \neq 0$, *and* $\sigma_{m,\varpi_i}(\pi) = \pi_{w_1\varpi_i} * \pi_{w_2\varpi_i} * \cdots * \pi_{w_m\varpi_i}$.

(2) *The converse of* (1) *holds. Namely, let* $\pi = x_{j_1}x_{j_2}\cdots x_{j_k}\pi_{\varpi_i} \in \mathbb{B}(\varpi_i)$. *Take* $m \in \mathbb{Z}_{>0}$ *such that the assertion of Proposition 3.1.3 holds, and assume that* $\sigma_{m,\varpi_i}(\pi) = \pi_{w_1\varpi_i} * \pi_{w_2\varpi_i} * \cdots * \pi_{w_m\varpi_i}$. *Then we have* $b := x_{j_1}x_{j_2}\cdots x_{j_k}u_{\varpi_i} \neq 0$, *and* $\sigma_{m,\varpi_i}(b) = u_{w_1\varpi_i} \otimes u_{w_2\varpi_i} \otimes \cdots \otimes u_{w_m\varpi_i}$.

4 Main Results.

4.1 Isomorphism theorem. From now on, we assume that $\mathfrak{g}$ is an affine Lie algebra. We can carry out the proof of our isomorphism theorem, following the general line of that for [Kas5, Theorem 4.1].

Theorem 4.1.1. *There exists a unique isomorphism* $\Phi_{\varpi_i} : \mathcal{B}(\varpi_i) \xrightarrow{\sim} \mathbb{B}_0(\varpi_i)$ *of crystals such that* $\Phi_{\varpi_i}(u_{\varpi_i}) = \pi_{\varpi_i}$.

Proof. It suffices to prove that for $j_1, j_2, \ldots, j_p \in I$ and $k_1, k_2, \ldots, k_q \in I$,

(1) $x_{j_1}x_{j_2}\cdots x_{j_p}u_{\varpi_i} = x_{k_1}x_{k_2}\cdots x_{k_q}u_{\varpi_i} \Leftrightarrow x_{j_1}x_{j_2}\cdots x_{j_p}\pi_{\varpi_i} = x_{k_1}x_{k_2}\cdots x_{k_q}\pi_{\varpi_i}$,

(2) $x_{j_1}x_{j_2}\cdots x_{j_p}u_{\varpi_i} = 0 \Leftrightarrow x_{j_1}x_{j_2}\cdots x_{j_p}\pi_{\varpi_i} = 0$.

Part (2) has already been proved in Lemma 3.2.2. Let us show the direction ($\Rightarrow$) of part (1). Take $m \in \mathbb{Z}_{>0}$ such that the assertion of Proposition 2.2.5 holds for both $b_1 := x_{j_1}x_{j_2}\cdots x_{j_p}u_{\varpi_i}$ and $b_2 := x_{k_1}x_{k_2}\cdots x_{k_q}u_{\varpi_i}$:

$$\sigma_{m,\varpi_i}(b_1) = u_{w_1\varpi_i} \otimes u_{w_2\varpi_i} \otimes \cdots \otimes u_{w_m\varpi_i},$$
$$\sigma_{m,\varpi_i}(b_2) = u_{w'_1\varpi_i} \otimes u_{w'_2\varpi_i} \otimes \cdots \otimes u_{w'_m\varpi_i}.$$

Since $b_1 = b_2$, we get $u_{w_l\varpi_i} = u_{w'_l\varpi_i}$, and hence $w_l\varpi_i = w'_l\varpi_i$ for all $l = 1, 2, \ldots, m$. By Lemma 3.2.2 (1), we see that

$$\sigma_{m,\varpi_i}(\pi_1) = \pi_{w_1\varpi_i} * \pi_{w_2\varpi_i} * \cdots * \pi_{w_m\varpi_i},$$
$$\sigma_{m,\varpi_i}(\pi_2) = \pi_{w'_1\varpi_i} * \pi_{w'_2\varpi_i} * \cdots * \pi_{w'_m\varpi_i},$$

where $\pi_1 := x_{j_1}x_{j_2}\cdots x_{j_p}\pi_{\varpi_i}$ and $\pi_2 := x_{k_1}x_{k_2}\cdots x_{k_q}\pi_{\varpi_i}$. Since $w_l\varpi_i = w'_l\varpi_i$ and $\pi_{w\varpi_i}(t) = t(w\varpi_i)$ for all $w \in W$, we get $\sigma_{m,\varpi_i}(\pi_1) = \sigma_{m,\varpi_i}(\pi_2)$. Since σ_{m,ϖ_i} is injective, we conclude that $\pi_1 = \pi_2$.

We show the reverse direction ($\Leftarrow$) of part (1). Take $m \in \mathbb{Z}_{>0}$ such that the assertion of Proposition 3.1.3 holds for both $\pi_1 := x_{j_1}x_{j_2}\cdots x_{j_p}\pi_{\varpi_i}$ and $\pi_2 :=$

$x_{k_1}x_{k_2}\cdots x_{k_q}\pi_{\varpi_i}$:

$$\sigma_{m,\varpi_i}(\pi_1) = \pi_{w_1\varpi_i} * \pi_{w_2\varpi_i} * \cdots * \pi_{w_m\varpi_i},$$

$$\sigma_{m,\varpi_i}(\pi_2) = \pi_{w'_1\varpi_i} * \pi_{w'_2\varpi_i} * \cdots * \pi_{w'_m\varpi_i}.$$

Since $\pi_1 = \pi_2$, and hence $\sigma_{m,\varpi_i}(\pi_1) = \sigma_{m,\varpi_i}(\pi_2)$ in $\mathbb{P}$, the two paths $\pi_{w_1\varpi_i} * \pi_{w_2\varpi_i} * \cdots * \pi_{w_m\varpi_i}$ and $\pi_{w'_1\varpi_i} * \pi_{w'_2\varpi_i} * \cdots * \pi_{w'_m\varpi_i}$ are identical modulo reparametrization. Hence we can deduce that $w_l\varpi_i = w'_l\varpi_i$ for all $l = 1, 2, \ldots, m$ from the fact that if $a\varpi_j \in W\varpi_i$ for some $a \in \mathbb{Q}_{\geq 0}$ and $i, j \in I_0$, then $i = j$ and $a = 1$. By Lemma 3.2.2 (2), we have

$$\sigma_{m,\varpi_i}(b_1) = u_{w_1\varpi_i} \otimes u_{w_2\varpi_i} \otimes \cdots \otimes u_{w_m\varpi_i},$$

$$\sigma_{m,\varpi_i}(b_2) = u_{w'_1\varpi_i} \otimes u_{w'_2\varpi_i} \otimes \cdots \otimes u_{w'_m\varpi_i}.$$

Since $w_l\varpi_i = w'_l\varpi_i$ for all $l = 1, 2, \ldots, m$, it follows from [Kas5, Proposition 5.8 (i)] that $u_{w_l\varpi_i} = u_{w'_l\varpi_i}$ for all $l = 1, 2, \ldots, m$. Therefore we have $\sigma_{m,\varpi_i}(b_1) = \sigma_{m,\varpi_i}(b_2)$. Since σ_{m,ϖ_i} is injective, we conclude that $b_1 = b_2$. □

Remark 4.1.2. In general, an isomorphism of crystals between $\mathcal{B}(\lambda)$ and $\mathbb{B}_0(\lambda)$ does not exist, even if $\mathcal{B}(\lambda)$ is connected. For example, let $\mathfrak{g}$ be of type $A_2^{(1)}$, and $\lambda = \varpi_1 + \varpi_2$ (we know from [Kas5, Proposition 5.4] that $\mathcal{B}(\lambda)$ is connected). If $\mathcal{B}(\lambda) \cong \mathbb{B}_0(\lambda)$ as crystals, then we would have $wu_\lambda = w'u_\lambda$ in $\mathcal{B}(\lambda)$ for every $w, w' \in W$ with $w\lambda = w'\lambda$, but we have an example of $w, w' \in W$ such that $wu_\lambda \neq w'u_\lambda$ in $\mathcal{B}(\lambda)$ and $w\lambda = w'\lambda$ (see [Kas5, Remark 5.10]).

Remark 4.1.3. In [G], Greenstein proved that if $\mathfrak{g}$ is of type $A_\ell^{(1)}$, then the connected component $\mathbb{B}_0(m\varpi_i+n\delta)$ is a path model for a certain bounded module $L(\ell, m, n)$. He also showed a decomposition rule for tensor products, which seems to be closely related to Theorem 4.3.3 below.

4.2 Branching rule for $V(\varpi_i)$.

Lemma 4.2.1. *For every $\pi \in \mathbb{B}(\varpi_i)$, we have $(\pi(1), \pi(1)) \leq (\varpi_i, \varpi_i)$.*

Proof. Let $\pi = (\nu_1, \nu_2, \ldots, \nu_s\,;\, a_0, a_1, \ldots, a_s)$ with $\nu_j \in W\varpi_i$ and $a_j \in [0, 1]$ be a Lakshmibai-Seshadri path of shape ϖ_i (cf. [L2, §4]). By the definition of a Lakshmibai–Seshadri path, we see that $\pi(1) = \sum_{j=1}^{s}(a_j - a_{j-1})\nu_j$. Hence we have

$$\begin{aligned}(\pi(1), \pi(1)) &= \sum_{j=1}^{s}(a_j - a_{j-1})^2(\nu_j, \nu_j) + 2\sum_{1\leq k<l\leq s}(a_k - a_{k-1})(a_l - a_{l-1})(\nu_k, \nu_l)\\ &= \sum_{j=1}^{s}(a_j - a_{j-1})^2(\varpi_i, \varpi_i) + 2\sum_{1\leq k<l\leq s}(a_k - a_{k-1})(a_l - a_{l-1})(\varpi_i, w_{kl}\varpi_i)\end{aligned}$$

for some $w_{kl} \in W$. By [Kac, Proposition 6.3], we deduce that $w_{kl}\varpi_i = \varpi_i - \beta_{kl} + n_{kl}\delta$ for some $\beta_{kl} \in \sum_{i \in I_0} \mathbb{Z}_{\geq 0}\alpha_i$ and $n_{kl} \in \mathbb{Z}$. Therefore, we have (note that ϖ_i is of level 0)

$$
\begin{aligned}
(\pi(1), \pi(1)) &= \sum_{j=1}^{s}(a_j - a_{j-1})^2(\varpi_i, \varpi_i) \\
&\qquad + 2 \sum_{1 \leq k < l \leq s} (a_k - a_{k-1})(a_l - a_{l-1})(\varpi_i, \varpi_i - \beta_{kl} + n_{kl}\delta) \\
&= \sum_{j=1}^{s}(a_j - a_{j-1})^2(\varpi_i, \varpi_i) + 2 \sum_{1 \leq k < l \leq s} (a_k - a_{k-1})(a_l - a_{l-1})(\varpi_i, \varpi_i) \\
&\qquad - 2 \sum_{1 \leq k < l \leq s} (a_k - a_{k-1})(a_l - a_{l-1})(\varpi_i, \beta_{kl}) \\
&= \left\{\sum_{j=1}^{s}(a_j - a_{j-1})\right\}^2 (\varpi_i, \varpi_i) - 2 \sum_{1 \leq k < l \leq s} (a_k - a_{k-1})(a_l - a_{l-1})(\varpi_i, \beta_{kl}) \\
&= (\varpi_i, \varpi_i) - 2 \sum_{1 \leq k < l \leq s} (a_k - a_{k-1})(a_l - a_{l-1})(\varpi_i, \beta_{kl}).
\end{aligned}
$$

Since $(\varpi_i, \beta_{kl}) \geq 0$ for all $1 \leq k < l \leq s$, we deduce that $(\pi(1), \pi(1)) \leq (\varpi_i, \varpi_i)$, as desired. □

Let S be a proper subset of I, i.e., $S \subsetneq I$. Let $\mathfrak{g}_S$ be the Levi subalgebra of $\mathfrak{g}$ corresponding to S, and $U_q(\mathfrak{g}_S) \subset U_q(\mathfrak{g})$ the quantized universal enveloping algebra of $\mathfrak{g}_S$. Note that a crystal for $U_q(\mathfrak{g})$ can be regarded as a crystal for $U_q(\mathfrak{g}_S)$ by restriction.

Theorem 4.2.2. *As crystals for $\mathfrak{g}_S$, $\mathbb{B}(\varpi_i)$ and $\mathbb{B}_0(\varpi_i)$ decompose as follows :*

$$
\mathbb{B}(\varpi_i) \cong \bigsqcup_{\substack{\pi \in \mathbb{B}(\varpi_i) \\ \pi:\ \mathfrak{g}_S\text{-dominant}}} \mathbb{B}_S(\pi(1)), \qquad \mathbb{B}_0(\varpi_i) \cong \bigsqcup_{\substack{\pi \in \mathbb{B}_0(\varpi_i) \\ \pi:\ \mathfrak{g}_S\text{-dominant}}} \mathbb{B}_S(\pi(1)), \tag{4.2.1}
$$

where $\mathbb{B}_S(\lambda)$ is the set of Lakshmibai–Seshadri paths of shape λ for $U_q(\mathfrak{g}_S)$, and a path π is said to be $\mathfrak{g}_S$-dominant if $(\pi(t))(\alpha_i^\vee) \geq 0$ for all $t \in [0,1]$ and $i \in S$.

Proof. We will show only the first equality in (4.2.1), since the second one can be shown in the same way. As in [Kas1, §9.3], we deduce, using Lemma 4.2.1, that each connected component of $\mathbb{B}(\varpi_i)$ (as a crystal for $U_q(\mathfrak{g}_S)$) contains an extremal weight element π' with respect to $W_S := \langle r_j \mid j \in S \rangle$. Because $\mathfrak{g}_S$ is a finite-dimensional reductive Lie algebra, there exists $w \in W_S$ such that $((w\pi')(1))(\alpha_j^\vee) \geq 0$ for all $j \in S$. Put $\pi := w\pi'$ for this $w \in W_S$. Since π is also extremal, we have that $e_j\pi = 0$ for all $j \in S$. Because π is a Lakshmibai–Seshadri path of shape

ϖ_i, we deduce from [L2, Lemmas 2.2 b) and 4.5 d)] that $(\pi(t))(\alpha_j^\vee) \geq 0$ for all $t \in [0,1]$ and $j \in S$, i.e., π is $\mathfrak{g}_S$-dominant. We see from [L2, Theorem 7.1] that the connected component containing π as a crystal for $U_q(\mathfrak{g}_S)$ is isomorphic to $\mathbb{B}_S(\pi(1))$, thereby completing the proof of the theorem. □

Theorem 4.2.3. (1) *The extremal weight module $V(\varpi_i)$ of extremal weight ϖ_i is completely reducible as a $U_q(\mathfrak{g}_S)$-module.*
(2) *The decomposition of $V(\varpi_i)$ as a $U_q(\mathfrak{g}_S)$-module is given by :*

$$V(\varpi_i) \cong \bigoplus_{\substack{\pi\in\mathbb{B}_0(\varpi_i) \\ \pi:\ \mathfrak{g}_S\text{-dominant}}} V_S(\pi(1)), \tag{4.2.2}$$

where $V_S(\lambda)$ is the integrable highest weight $U_q(\mathfrak{g}_S)$-module of highest weight λ.

Proof. (1) First we prove that $U := U_q(\mathfrak{g}_S)u$ is finite-dimensional for each weight vector $u \in V(\varpi_i)$. To prove this, it suffices to show that the weight system $\mathrm{Wt}(U)$ of U is a finite set, since each weight space of $V(\varpi_i)$ is finite-dimensional (see [Kas5, Proposition 5.16 (iii)]). Remark that if μ, $\nu \in P$ are weights of U, then μ, $\nu \in \mathfrak{h}_0^*$, and $\mu - \nu \in Q_S := \sum_{i\in S}\mathbb{Z}\alpha_i$. Hence the canonical map $\mathrm{cl} : \mathfrak{h}_0^* \twoheadrightarrow \mathfrak{h}_0^*/\mathbb{Q}\delta$ is injective on $\mathrm{Wt}(U)$, since $k\delta \notin Q_S$ for any $k \in \mathbb{Z}\setminus\{0\}$. Since $\mathrm{Wt}(U)$ is contained in the weight system $\mathrm{Wt}(V(\varpi_i))$ of $V(\varpi_i)$, it follows from Theorem 4.1.1 and Lemma 4.2.1 that

$$\begin{aligned}\mathrm{cl}(\mathrm{Wt}(U)) \subset \mathrm{cl}\big(\mathrm{Wt}(V(\varpi_i))\big) &= \mathrm{cl}\big(\{\pi(1) \mid \pi \in \mathbb{B}_0(\varpi_i)\}\big) \quad \text{by Theorem 4.1.1}\\ &\subset \big\{\mu' \in \mathfrak{h}_0^*/\mathbb{Q}\delta \mid (\mu',\mu') \leq (\mathrm{cl}(\varpi_i),\mathrm{cl}(\varpi_i))\big\} \quad \text{by Lemma 4.2.1.}\end{aligned}$$

Because the bilinear form $(\cdot\,,\,\cdot)$ on $\mathfrak{h}_0^*/\mathbb{Q}\delta$ is positive-definite, the set $\mathrm{cl}(\mathrm{Wt}(U))$ is discrete and contained in a compact set with respect to the usual metric topology on $\mathbb{R}\otimes_{\mathbb{Q}}(\mathfrak{h}_0^*/\mathbb{Q}\delta)$ defined by $(\cdot\,,\,\cdot)$. Therefore, we see that $\mathrm{cl}(\mathrm{Wt}(U))$ is a finite set, and hence so is $\mathrm{Wt}(U)$. Thus, we conclude that $U = U_q(\mathfrak{g}_S)u$ is finite-dimensional.

Since q is assumed to generic, the finite-dimensional $U_q(\mathfrak{g}_S)$-module $U_q(\mathfrak{g}_S)u$ is completely reducible for each weight vector $u \in V(\varpi_i)$. Because $V(\varpi_i)$ is a sum of all such modules $U_q(\mathfrak{g}_S)u$, we deduce that $V(\varpi_i)$ is also completely reducible.

(2) Because each weight space of $V(\varpi_i)$ is finite-dimensional, we can define the formal character $\mathrm{ch}\,V(\varpi_i)$ of $V(\varpi_i)$. By Theorem 4.2.2, we have

$$\mathrm{ch}\,V(\varpi_i) = \sum_{\substack{\pi\in\mathbb{B}_0(\varpi_i) \\ \pi:\ \mathfrak{g}_S\text{-dominant}}} \mathrm{ch}\,V_S(\pi(1)).$$

Therefore, in order to prove part (2), we need only show that this is the unique way of writing $\operatorname{ch} V(\varpi_i)$ as a sum of the characters of integrable highest weight $U_q(\mathfrak{g}_S)$-modules. Assume that

$$\operatorname{ch} V(\varpi_i) = \sum_{\lambda \in P} c_\lambda \operatorname{ch} V_S(\lambda) \quad \text{and} \quad \operatorname{ch} V(\varpi_i) = \sum_{\lambda \in P} c'_\lambda \operatorname{ch} V_S(\lambda)$$

with $c_\lambda, c'_\lambda \in \mathbb{Z}$ for $\lambda \in P$. Then we have $\sum_{\lambda \in P}(c_\lambda - c'_\lambda) \operatorname{ch} V_S(\lambda) = 0$. Suppose that there exists $\lambda \in P$ such that $c_\lambda - c'_\lambda \neq 0$, and set $X := \{\lambda \in P \mid c_\lambda - c'_\lambda \neq 0\} (\neq \emptyset)$. Note that X is contained in the weight system $\operatorname{Wt}(V(\varpi_i))$ of $V(\varpi_i)$. As in the proof of part (1), we deduce that

$$\operatorname{cl}\bigl(\operatorname{Wt}(V(\varpi_i))\bigr) \subset \{\mu' \in \mathfrak{h}_0^*/\mathbb{Q}\delta \mid (\mu', \mu') \leq (\operatorname{cl}(\varpi_i), \operatorname{cl}(\varpi_i))\},$$

and hence $\operatorname{Wt}(V(\varpi_i))$ modulo $\mathbb{Z}\delta$ is a finite set.

Now, we define a partial order $\geq_S$ on P as follows:

$$\mu \geq_S \nu \quad \text{for} \quad \mu, \nu \in P \quad \Longleftrightarrow \quad \mu - \nu \in (Q_S)_+ := \sum_{i \in S} \mathbb{Z}_{\geq 0} \alpha_i.$$

Let us show that the set X has a maximal element with respect to this order $\geq_S$. Let $\mu \in X$. Then $\operatorname{Wt}(V(\varpi_i)) \cap (\mu + Q_S)$ is a finite set. Indeed, if this is not a finite set, then there exist elements ν, ν' of it such that $\nu - \nu' = k\delta$ with $k \in \mathbb{Z} \setminus \{0\}$, since $\operatorname{Wt}(V(\varpi_i))$ modulo $\mathbb{Z}\delta$ is a finite set. However, since $\nu - \nu' \in Q_S$ and $k\delta \notin Q_S$ for any $k \in \mathbb{Z} \setminus \{0\}$, this is a contradiction. Therefore, we see that $X \cap (\mu + (Q_S)_+)$ is also a finite set, and hence that X has a maximal element of the form $\mu + \beta$ for some $\beta \in (Q_S)_+$.

Let $\nu \in X$ be a maximal element with respect to this order $\geq_S$. We can easily see that the coefficient of $e(\nu)$ in $\sum_{\lambda \in P}(c_\lambda - c'_\lambda) \operatorname{ch} V_S(\lambda)$ is equal to $c_\nu - c'_\nu$. Since $\nu \in X$, we have $c_\nu - c'_\nu \neq 0$, which contradicts $\sum_\lambda (c_\lambda - c'_\lambda) \operatorname{ch} V_S(\lambda) = 0$. This completes the proof of the theorem. □

4.3 Decomposition rule for tensor products. In this subsection, we assume that ϖ_i is minuscule, i.e., $\varpi_i(\alpha^\vee) \in \{\pm 1, 0\}$ for every dual real root $\alpha^\vee$ of $\mathfrak{g}$.

Remark 4.3.1. The following is the list of minuscule weights (cf. [H, p. 174]). We use the numbering of vertices of the Dynkin diagrams in [Kac, Ch. 4]:

$A_\ell^{(1)}\ (\ell \geq 1):$	$\varpi_1,\ \varpi_2,\ \ldots,\ \varpi_\ell$	$A_{2\ell-1}^{(2)}\ (\ell \geq 3):$	ϖ_1
$B_\ell^{(1)}\ (\ell \geq 3):$	ϖ_ℓ	$D_{\ell+1}^{(2)}\ (\ell \geq 2):$	ϖ_ℓ
$C_\ell^{(1)}\ (\ell \geq 2):$	ϖ_1		
$D_\ell^{(1)}\ (\ell \geq 4):$	$\varpi_1,\ \varpi_{\ell-1},\ \varpi_\ell$		
$E_6^{(1)}:$	$\varpi_1,\ \varpi_5$		
$E_7^{(1)}:$	ϖ_6		

Remark 4.3.2. If ϖ_i is minuscule, then, for any $\mu,\ \nu \in W\varpi_i$ and rational number $0 < a < 1$, there does not exist an a-chain for (μ, ν). Hence it follows from the definition of Lakshmibai–Seshadri paths that $\mathbb{B}(\varpi_i) = \{\pi_{w\varpi_i} \mid w \in W\}$. Since $w\pi_{\varpi_i} = \pi_{w\varpi_i}$, we see that $\mathbb{B}(\varpi_i)$ is connected, and hence $\mathbb{B}(\varpi_i) = \mathbb{B}_0(\varpi_i)$.

Theorem 4.3.3. *Let λ be a dominant integral weight which is not a multiple of the null root δ of $\mathfrak{g}$. Then, the concatenation $\mathbb{B}(\lambda) * \mathbb{B}(\varpi_i)$ decomposes as follows:*

$$\mathbb{B}(\lambda) * \mathbb{B}(\varpi_i) \cong \bigsqcup_{\substack{\pi \in \mathbb{B}(\varpi_i) \\ \pi:\ \lambda\text{-dominant}}} \mathbb{B}(\lambda + \pi(1)), \tag{4.3.1}$$

where $\pi \in \mathbb{B}(\varpi_i)$ is said to be λ-dominant if $(\lambda + \pi(t))(\alpha_i^\vee) \geq 0$ for all $t \in [0, 1]$ and $i \in I$.

Proof. We will prove that each connected component contains a (unique) path of the form $\pi_\lambda * \pi$ for a λ-dominant path $\pi \in \mathbb{B}(\varpi_i)$. Then the assertion of the theorem follows from [L2, Theorem 7.1].

Let $\pi_1 * \pi_2 \in \mathbb{B}(\lambda) * \mathbb{B}(\varpi_i)$. It can easily be seen that $e_{i_1} e_{i_2} \cdots e_{i_k}(\pi_1 * \pi_2) = \pi_\lambda * \pi_2'$ for some $i_1,\ i_2,\ \ldots,\ i_k \in I$, where $\pi_2' \in \mathbb{B}(\varpi_i)$ (cf. [G, §5.6]). Set $S := \{i \in I \mid \lambda(\alpha_i^\vee) = 0\}$ (note that $S \subsetneq I$, since λ is not a multiple of δ), and let $\mathbb{B}$ be the set of paths of the form $e_{j_1} e_{j_2} \cdots e_{j_l}(\pi_\lambda * \pi_2')$ for $j_1,\ j_2,\ \ldots,\ j_l \in S$. Remark that if $e_{j_1} e_{j_2} \cdots e_{j_l}(\pi_\lambda * \pi_2') \neq 0$, then $e_{j_1} e_{j_2} \cdots e_{j_l}(\pi_\lambda * \pi_2') = \pi_\lambda * (e_{j_1} e_{j_2} \cdots e_{j_l} \pi_2')$. As in the proof of part (2) of Theorem 4.2.3, we deduce that

$$\{\pi(1) \mid \pi \in \mathbb{B}(\varpi_i)\} \cap (\pi_2'(1) + (Q_S)_+) = \mathrm{Wt}(V(\varpi_i)) \cap (\pi_2'(1) + (Q_S)_+)$$

is a finite set. Hence we have $\pi_\lambda * \pi_2'' \in \mathbb{B}$ for some $\pi_2'' \in \mathbb{B}(\varpi_i)$ such that $e_j(\pi_\lambda * \pi_2'') = 0$ for all $j \in S$. Because ϖ_i is minuscule and $\pi_2'' = \pi_{w\varpi_i}$ for some $w \in W$ (cf. Remark 4.3.2), we see that $e_j(\pi_\lambda * \pi_2'') = 0$ for all $j \in I \setminus S$. Therefore, we conclude that $\pi_2'' \in \mathbb{B}(\varpi_i)$ is λ-dominant. Thus, we have completed the proof of the theorem. □

Remark 4.3.4. Unlike Theorems 4.2.2 and 4.2.3, this theorem does not necessarily imply the decomposition rule for tensor products of corresponding $U_q(\mathfrak{g})$-modules.

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